

# Identifying of the refractive index for the acoustic equation at fixed frequency

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## Abstract

In this paper we determine a formula for calculating the refractive index  $\mathbf{n}$  for the acoustic equation from the partial Dirichlet to Neumann map(DN) associated to  $\mathbf{n}$ . We apply these results to identify locations and values of small volume perturbations of this refractive index at fixed frequency  $\omega$ .

**Identification de l'indice de réfraction de l'équation acoustique à fréquence fixe**

## Résumé

Dans cette Note nous déterminons une formule pour calculer l'indice de réfraction  $\mathbf{n}$  pour l'équation acoustique à partir de l'application Dirichlet-to-Neumann partielle (DN) associée à  $\mathbf{n}$ . Nous nous appliquerons ces résultats pour identifier les locations et les valeurs des petites perturbations volumiques associées à cet indice de réfraction pour une fréquence  $\omega$  fixe.

## Version française abrégée

Cette Note traite un problème inverse pour l'équation acoustique avec fréquence fixe. Le but est d'identifier l'indice de réfraction associé après avoir dériver une formule appropriée à l'aide de l'application Dirichlet-to-Neumann partielle (DN) associée à  $\mathbf{n}$  et de donner plus explicitement cette identification. Concernant l'utilisation de l'application Dirichlet-to-Neumann partielle en problème d'identification, nous pouvons trouver, par exemple, le travail De Kohn et Vogelius [5].

Soit  $\Omega \subset \mathbf{R}^3$  un domaine borné avec un bord de classe  $C^2$ . Nous noterons  $\nu$  la normale unitaire sortante du bord  $\partial\Omega$ . Soit  $\Gamma$  une partie ouverte régulière de la frontière  $\partial\Omega$ . Supposons que  $\Omega$  contient un nombre fini d'inhomogénéités, chacune de la forme  $z_j + \alpha B_j$ , où  $B_j \subset \mathbf{R}^3$  bornée et contenant l'origine.

Soit  $\mathbf{n}(x) \in C^0(\Omega)$  l'indice de réfraction non perturbé. Nous supposons que  $\mathbf{n}(x)$  est connue sur un voisinage de  $\partial\Omega$ . Notons par  $\mathbf{n}_j(x) \in C^0(\overline{z_j + \alpha B_j})$  l'indice de réfraction de la  $j$ -ème inhomogénéité,  $z_j + \alpha B_j$ .

Considérons l'équation acoustique, à fréquence fixe, en présence d'inhomogénéités

$$(\Delta + \omega^2 \mathbf{n}_\alpha) u_\alpha = 0 \text{ et } \Omega$$

$$u_\alpha|_{\partial\Omega} = f \in \tilde{H}^{\frac{1}{2}}(\Gamma),$$

et définissons l'application Dirichlet-to-Neumann partielle associée à  $\mathbf{n}_\alpha$  par:  $\Lambda_{\mathbf{n}_\alpha}(f) = \frac{\partial u_\alpha}{\partial \nu}|_\Gamma$  pour tous  $f \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ . Ici  $\tilde{H}^{\frac{1}{2}}(\Gamma)$  désigne l'espace trace.

Bukhgeim et Uhlmann [4] ont utilisé les solutions complexes de l'optique géométrique pour prouver que la connaissance des données partielles de Cauchy pour l'équation de Schrödinger

détermine le potentiel de manière unique sachant que le potentiel  $q \in L^\infty(\Omega)$  est connue dans un voisinage du bord. Nous pouvons utiliser leurs travaux pour montrer notre version physique et nous utilisons ainsi ces solutions complexes de l'optique géométrique construites dans leur travaux pour prouver notre méthode de reconstruction.

Ensuite, nous dérivons une formule de calcul de l'indice de réfraction  $\mathbf{n}$  à partir de l'application Dirichlet-to-Neumann partielle associé à  $\mathbf{n}$  et  $\Gamma$  en modifiant la procédure de reconstruction de Nachman [6]. Enfin, nous profitons des propriétés des noyaux des opérateurs ainsi introduites pour présenter plus explicitement cette formule de reconstruction. Nous considérons le problème inverse d'identification des lieux de petites perturbations volumique de l'indice de réfraction et nous appliquons nos résultats pour réduire ce problème inverse au calcul de transformation de Fourier inverse.

## 1 Problem formulation

The paper deals with an inverse problem for the acoustic equation with fixed frequency. The object is to identify the associated refractive index after deriving a suitable formula by using the partial Dirichlet to Neumann map(DN) associated to  $\mathbf{n}$  and to give more explicitly this identification. Concerning the use of the partial Dirichlet to Neumann map in recovering problem we can find, for example, the work of Kohn and Vogelius [5].

Let  $\Omega \subset \mathbf{R}^3$  be a bounded domain with  $C^2$  boundary. We denote by  $\nu$  the unit-outer normal to  $\partial\Omega$ . Let  $\Gamma$  be a smooth open subset of the boundary  $\partial\Omega$  and  $\Gamma_c$  denotes  $\partial\Omega \setminus \overline{\Gamma}$ . We suppose throughout that  $\mathbf{R}^3 \setminus \overline{\Omega}$ . Introduce the trace space

$$\tilde{H}^{\frac{1}{2}}(\Gamma) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega), f \equiv 0 \text{ on } \Gamma_c \right\}.$$

Here and in the sequel we identify  $f$  defined only on  $\Gamma$  with its extension by 0 to all  $\partial\Omega$ . It is known that the dual of  $\tilde{H}^{\frac{1}{2}}(\Gamma)$  is  $H^{-\frac{1}{2}}(\Gamma)$ .

Assume that  $\Omega$  contains a finite number of inhomogeneities, each of the form  $z_j + \alpha B_j$ , where  $B_j \subset \mathbf{R}^3$  is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is  $\mathcal{B}_\alpha = \cup_{j=1}^m (z_j + \alpha B_j)$ . The points  $z_j \in \Omega, j = 1, \dots, m$ , which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

$$(1) \quad |z_j - z_l| \geq c_0 > 0, \forall j \neq l \quad \text{and} \quad \text{dist}(z_j, \partial\Omega) \geq c_0 > 0, \forall j,$$

where  $c_0$  is a positive constant. Assume that  $\alpha > 0$ , the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to  $\mathbf{R}^3 \setminus \overline{\Omega}$  is larger than  $c_0$ .

Let  $\mathbf{n}(x) \in C^0(\Omega)$  denote the unperturbed refractive index. We assume that  $\mathbf{n}(x)$  is known on a neighborhood of the boundary  $\partial\Omega$ . Let  $\mathbf{n}_j(x) \in C^0(\overline{z_j + \alpha B_j})$  denote the refractive index of the  $j$ -th inhomogeneity,  $z_j + \alpha B_j$ . Introduce the perturbed potential

$$(2) \quad \mathbf{n}_\alpha(x) = \begin{cases} \mathbf{n}(x), & x \in \Omega \setminus \overline{\mathcal{B}_\alpha}, \\ \mathbf{n}_j(x), & x \in z_j + \alpha B_j, j = 1 \dots m. \end{cases}$$

Consider the acoustic equation, at fixed frequency, in the presence of the inhomogeneities  $\mathcal{B}_\alpha$

$$\begin{aligned} (\Delta + \omega^2 \mathbf{n}_\alpha) u_\alpha &= 0 \text{ in } \Omega \\ u_\alpha|_{\partial\Omega} &= f \in \tilde{H}^{\frac{1}{2}}(\Gamma), \end{aligned}$$

and define the local Dirichlet to Neumann map associated to  $\mathbf{n}_\alpha$  by  $\Lambda_{\mathbf{n}_\alpha}(f) = \frac{\partial u_\alpha}{\partial \nu}|_\Gamma$  for all  $f \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ . Where  $\tilde{H}^{\frac{1}{2}}(\Gamma)$  is the trace space

Bukhgeim and Uhlmann [4] used the complex geometrical optics solutions vanishing on the complementary of  $\Gamma$  to prove that the knowledge of the partial Cauchy data for the Schrödinger

equation on any open subset  $\Gamma$  of the boundary determines uniquely the potential  $q$  provided that  $q \in L^\infty(\Omega)$  is known in a neighborhood of the boundary. We may refer to their work to show our physical version and we use these complex geometrical optics solutions, constructed in their work, to prove our reconstruction procedure.

Next, we derive a formula for calculating the refractive index  $\mathbf{n}$  from the partial Dirichlet-to-Neumann map (DN) associated to  $\mathbf{n}$  and  $\Gamma$  by modifying the Nachman's reconstruction procedure [6]. It turns out that if the refractive index is a-priori known in a neighborhood of the boundary the derivation of a reconstruction formula are *surprisingly simple*. Finally, we profit to properties of the kernel of the introduced operator to explicit more this reconstruction formula. We consider the inverse problem of identifying locations of small volume fraction perturbations of the refractive index and we apply our results to reduce this inverse problem to calculations of inverse Fourier transforms.

## 2 Identification procedure

Let  $\mathbf{n}_0 \in L^\infty(\Omega)$  be a known function. Assume that  $\mathbf{n} = \mathbf{n}_0$  almost everywhere in a neighborhood of  $\partial\Omega$ . We extend  $\mathbf{n}$  and  $\mathbf{n}_0$  by 0 in  $\mathbf{R}^3$ . Let  $\rho \in \mathbf{R}^3 \setminus \{0\}$  and define  $u_\rho$  to be the solution of

$$(3) \quad \begin{aligned} \Delta u_\rho &= 0 && \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ (\frac{1}{\omega^2} \Delta + \mathbf{n})u_\rho &= 0 && \text{in } \Omega, \end{aligned}$$

we find that  $u_\rho|_\Gamma$  solves the (hyper-singular) integral equation on the open surface  $\Gamma$ :

$$(4) \quad \Lambda_{\mathbf{n}}(u_\rho|_\Gamma) + \oint_\Gamma \frac{\partial^2 g_\rho^D}{\partial \nu(x) \partial \nu(y)}(x, y) u_\rho|_\Gamma(y) ds(y) = \rho \cdot \nu(x) e^{x \cdot \rho}, \quad \forall x \in \Gamma,$$

where  $g_\rho^D$  is a well defined exterior Dirichlet Green's function for  $\Delta$ .

Now, according to [1] and [3] we can prove the following global uniqueness result associated to refractive index in absence of any inhomogeneities (in presence of the inhomogeneities we may obtain a similar result), as follows.

**Proposition 1** *Suppose that  $\alpha = 0$ . Let  $\mathbf{n}_i$  real-valued in  $L^\infty(\Omega)$ ,  $i = 1, 2$ . Assume  $\mathbf{n}_1 = \mathbf{n}_2$  almost everywhere in a neighborhood of the boundary  $\partial\Omega$  and  $\Lambda_{\mathbf{n}_1} = \Lambda_{\mathbf{n}_2}$ . Then  $\mathbf{n}_1 = \mathbf{n}_2$  almost everywhere in  $\Omega$ .*

*Proof.* Let  $-1 < \delta < 0$  and introduce the weighted  $L^2$ -space

$$L_\delta^2(\mathbf{R}^3) = \{f \in L_{loc}^2(\mathbf{R}^3) \mid \int_{\mathbf{R}^3} (1 + |x|^2)^\delta |f(x)|^2 dx < +\infty\}.$$

From [7], we know that if  $\tilde{\mathbf{n}}_i$  is an extant function of  $\mathbf{n}_i$  defined by:

$$\tilde{\mathbf{n}}_i(x) = \begin{cases} \mathbf{n}_i(x), & x \in \Omega, \\ 0, & x \in \mathbf{R}^3 \setminus \overline{\Omega}, \end{cases}$$

then the solutions of  $(\frac{1}{\omega^2} \Delta + \tilde{\mathbf{n}}_i)v_i = 0$  on  $\mathbf{R}^3$  can be given by:

$$(5) \quad v_i = e^{x \cdot \rho_i} (1 + \psi_{\mathbf{n}_i}(x, \rho_i)), i = 1, 2$$

for  $|\rho_i|$  sufficiently large with  $\psi_{\mathbf{n}_i}(\cdot, \rho_i) \in L_\delta^2(\mathbf{R}^3)$ . Moreover, we have

$$(6) \quad \|\psi_{\mathbf{n}_i}(\cdot, \rho_i)\|_{L_\delta^2(\mathbf{R}^3)} \leq \frac{C}{|\rho_i|}.$$

Now, we set

$$\rho_1 = \frac{\eta}{2} + i(\frac{k+l}{2}) \quad \text{and} \quad \rho_2 = -\frac{\eta}{2} + i(\frac{k-l}{2})$$

where  $\eta, k, l \in \mathbf{R}^3$  such that  $\eta \cdot k = \eta \cdot l = k \cdot l = 0$ , and  $|\eta|^2 = |k|^2 + |l|^2$ .  
Let  $\Omega' \subset \subset \Omega$ ,  $\Omega'$  open set with  $C^2$  boundary and  $\Omega \setminus \overline{\Omega'}$  is connected. Define

$$\tilde{N}(\Omega) = \left\{ v \in H^2(\Omega) \mid \left( \frac{1}{\omega^2} \Delta + \mathbf{n} \right) v = 0 \text{ in } \Omega, v = 0 \text{ on } \Gamma_c \right\}$$

and

$$N(\Omega) = \left\{ v \in H^2(\Omega) \mid \left( \frac{1}{\omega^2} \Delta + \mathbf{n} \right) v = 0 \text{ in } \Omega \right\}.$$

Then, according to [3] the set  $\tilde{N}(\Omega)$  is dense, in the  $L^2(\Omega')$  norm, in  $N(\Omega)$ . On the other hand, by Green's theorem we have

$$(7) \quad \int_{\Omega'} (\mathbf{n}_1 - \mathbf{n}_2) u_1 u_2 \, dx = \int_{\Gamma} \left( \frac{\partial u_1}{\partial \nu} u_2 - u_1 \frac{\partial u_2}{\partial \nu} \right) ds(x),$$

where  $ds(x)$  denotes surface measure and  $u_1, u_2 \in \tilde{N}(\Omega)$ . Let  $z_1 \in H^1(\Omega) \cap \tilde{N}(\Omega)$  such that  $u_2|_{\Gamma} = z_1|_{\Gamma}$ . The hypothesis  $\Lambda_{\mathbf{n}_1} = \Lambda_{\mathbf{n}_2}$  gives that

$$(8) \quad z_1|_{\Gamma_c} = 0, z_1|_{\Gamma} = u_2|_{\Gamma} \Rightarrow \frac{\partial z_1}{\partial \nu}|_{\Gamma} = \frac{\partial u_2}{\partial \nu}|_{\Gamma}.$$

Combining relation (8) with (7), we deduce:

$$\begin{aligned} \int_{\Omega'} (\mathbf{n}_1 - \mathbf{n}_2) u_1 u_2 \, dx &= \int_{\Gamma} \left( \frac{\partial u_1}{\partial \nu} u_2 - u_1 \frac{\partial u_2}{\partial \nu} \right) ds \\ &= \int_{\Gamma} \left( \frac{\partial u_1}{\partial \nu} z_1 - u_1 \frac{\partial z_1}{\partial \nu} \right) ds. \end{aligned}$$

By Green's theorem, we have:

$$\int_{\Gamma} \left( \frac{\partial u_1}{\partial \nu} z_1 - u_1 \frac{\partial z_1}{\partial \nu} \right) ds = \int_{\Omega'} (\mathbf{n}_1 - \mathbf{n}_2) u_1 z_1 \, dx = 0,$$

which implies

$$\int_{\Omega'} (\mathbf{n}_1 - \mathbf{n}_2) u_1 u_2 \, dx = 0.$$

Now, using density's argument we can approximate any  $z_i \in N(\Omega)$  by elements of  $\tilde{N}(\Omega)$ . Therefore the functions  $v_i$  defined in (5) and belong to  $N(\Omega)$  satisfy

$$\int_{\Omega'} (\mathbf{n}_1 - \mathbf{n}_2) z_1 z_2 \, dx = 0.$$

Letting  $|l| \rightarrow +\infty$ , and using the estimation (6), we deduce:

$$(\widehat{\mathbf{n}_1 - \mathbf{n}_2})(k) = 0 \quad \forall k \in \mathbf{R}^3$$

which achieves the proof. □

Next, define the double layer potential

$$N_{\rho}(f) = \oint_{\Gamma} \frac{\partial^2 g_{\rho}^D}{\partial \nu(x) \partial \nu(y)}(x, y) f|_{\Gamma}(y) \, ds(y), \quad \forall f \in \tilde{H}^{\frac{1}{2}}(\Gamma),$$

and set:

$$(9) \quad \begin{aligned} \rho_1 &= \frac{\eta}{2} + i \left( \frac{k+l}{2} \right) \\ \rho_2 &= -\frac{\eta}{2} + i \left( \frac{k-l}{2} \right) \end{aligned}$$

where  $\eta, k, l \in \mathbf{R}^3$  such that  $\eta \cdot k = \eta \cdot l = k \cdot l = 0$ , and  $|\eta|^2 = |k|^2 + |l|^2$ . The following holds.

**Lemma 1** Assume that 0 is not a Dirichlet eigenvalue of  $(\frac{1}{\omega^2}\Delta + \mathbf{n})$  in  $\Omega$  and  $\rho_1$  be given as in (9). Then, there is a unique solution  $u_{\rho_1}|_{\Gamma} \in \tilde{H}^{\frac{1}{2}}(\Gamma)$  of (4) such that

$$u_{\rho_1}|_{\Gamma} = (\Lambda_{\mathbf{n}} + N_{\rho_1})^{-1}(\rho_1 \cdot \nu(x)e^{x \cdot \rho_1}|_{\Gamma}).$$

Next, we can prove the following reconstruction formula by using Proposition 1.

**Proposition 2** Let  $\mathbf{n}_0 \in L^\infty(\Omega)$  be a given function. Assume that 0 is not a Dirichlet eigenvalue of  $(\frac{1}{\omega^2}\Delta + \mathbf{n})$  in  $\Omega$  and  $\mathbf{n} = \mathbf{n}_0$  almost everywhere in a neighborhood of  $\partial\Omega$ . Then

$$\begin{aligned} (\widehat{\mathbf{n} - \mathbf{n}_0})(-k) &= \frac{1}{\omega^2} \lim_{|l| \rightarrow +\infty} \int_{\Gamma} (\Lambda_{\mathbf{n}} + N_{\rho_1})^{-1}(\rho_1 \cdot \nu(x)e^{x \cdot \rho_1}|_{\Gamma})(\Lambda_{\mathbf{n}} - \\ &\quad \Lambda_{\mathbf{n}_0})(\Lambda_{\mathbf{n}_0} + N_{\rho_2})^{-1}(\rho_2 \cdot \nu(x)e^{x \cdot \rho_2}|_{\Gamma}) ds(x), \end{aligned}$$

where  $k, \rho_1$  and  $\rho_2$  are given as in (9).

*Proof.* Let  $\rho_i \in \mathbf{R}^3 \setminus \{0\}$  ( $i = 1, 2$ ). Let  $u_{\rho_i}|_{\Gamma} \in \tilde{H}^{\frac{1}{2}}(\Gamma)$  be the solution of (4) and define

$$\theta_{\mathbf{n}}(x, \rho_i) = e^{-x \cdot \rho_i} u_{\rho_i}(x) - 1,$$

then we have  $u_{\rho_i} = e^{-x \cdot \rho_i} (1 + \theta_{\mathbf{n}}(x, \rho_i))$ .

We decompose  $\rho_s = \tau(\xi + i\eta)$  ( $i^2 = -1$ ) with  $\xi, \eta \in \mathbf{R}^3$ ,  $|\xi| = |\eta| = 1$ . Then we can write:

$$\Delta_{\rho} \theta_{\mathbf{n}} = \mathbf{n} \chi(\Omega) \theta_{\mathbf{n}} \quad \text{in } \mathbf{R}^3 \setminus \Gamma_c,$$

where  $\chi(\Omega)$  is the characteristic function of  $\Omega$ . In a small neighborhood  $\mathcal{V}(\partial\Omega)$  of the boundary  $\partial\Omega$  let  $\nu_0$  denote the normal coordinate and  $\varphi(\frac{\nu_0}{|\rho|})$  a smooth cut-off function which vanishes on  $\mathcal{V}(\partial\Omega)$ . The fact that  $\Omega'$  is compactly supported in  $\Omega$  as mentioned in the proof of Proposition 1, we have  $\varphi(\frac{\nu_0}{|\rho|}) \theta_{\mathbf{n}} = \theta_{\mathbf{n}}$  on  $\Omega'$  for  $|\rho|$  sufficiently large. In order to get the aid of Proposition 1 we, firstly, demonstrate the following estimate

$$\|\theta_{\mathbf{n}}(\cdot, \rho)\|_{L^2(\Omega')} \leq \frac{C}{|\rho|}.$$

To do this, we may set  $\tilde{\theta}_{\mathbf{n}} = \varphi(\frac{\nu}{|\rho|}) \theta_{\mathbf{n}}$  in  $\mathbf{R}^3$  then

$$\Delta_{\rho} \tilde{\theta}_{\mathbf{n}} = \mathbf{n} \chi(\Omega) \tilde{\theta}_{\mathbf{n}} + \frac{1}{|\rho|} \nabla \cdot \tilde{\theta}_{\mathbf{n}} \cdot \nabla \varphi + \tilde{\theta}_{\mathbf{n}} \left( \frac{1}{|\rho|^2} + \Delta \varphi + 2 \frac{\rho}{|\rho|} \cdot \nabla \varphi \right) \quad \text{in } \mathbf{R}^3.$$

Obviously that  $\tilde{\theta}_{\mathbf{n}} \in L_{\delta}^2$ , therefore by classical results [7] the following estimate holds

$$(10) \quad \|\tilde{\theta}_{\mathbf{n}}\|_{L_{\delta}^2(\mathbf{R}^3)} \leq \frac{C}{|\tau|},$$

for some positive constant  $C$  independent of  $\tau$ . Next, as done in (9) we set

$$\begin{aligned} \rho_1 &= \frac{\eta}{2} + i \left( \frac{k+l}{2} \right) \\ \rho_2 &= -\frac{\eta}{2} + i \left( \frac{k-l}{2} \right) \end{aligned}$$

where  $\eta, k, l \in \mathbf{R}^3$  such that  $\eta \cdot k = \eta \cdot l = k \cdot l = 0$ , and  $|\eta|^2 = |k|^2 + |l|^2$ .

By applying last results, we construct

$$u = u_{\rho} = e^{x \cdot \rho_1} (1 + \theta_{\mathbf{n}}(x, \rho_1)) \quad \text{in } \Omega'$$

and

$$v = v_{\rho} = e^{x \cdot \rho_2} (1 + \theta_{\mathbf{n}_0}(x, \rho_2)) \quad \text{in } \Omega',$$

to obtain that

$$(11) \quad (\widehat{\mathbf{n} - \mathbf{n}_0})(-k) = \lim_{|l| \rightarrow +\infty} \int_{\Gamma} u_{\rho}(\Lambda_{\mathbf{n}} - \Lambda_{\mathbf{n}_0}) v_{\rho} ds.$$

Finally, by Lemma 1 one can see that the boundary values of the solutions  $u_{\rho}|_{\Gamma}$  can be recovered from  $\Lambda_{\mathbf{n}}$ . The result is then proven.  $\square$

The following lemma is useful to give more explanation to our reconstruction method.

**Lemma 2** *Let  $\rho_1$  and  $\rho_2$  be given as in (9), the following asymptotic behavior holds :*

$$N_{\rho_i} = |l|L_i + O\left(\frac{1}{|l|}\right) \text{ as } |l| \rightarrow +\infty,$$

where, for  $i \in \{1, 2\}$ ,  $L_i$  is a well defined integral operator on  $\tilde{H}^{\frac{1}{2}}(\Gamma)$ .

*Proof.* We give a brief proof for  $i = 1$ , and the case  $i = 2$  can be given similarly. Recall that the kernel of the operator  $N_{\rho_1}$  is  $\frac{\partial^2 g_{\rho_1}^D}{\partial \nu(x) \partial \nu(y)}(x, y)$  where  $g_{\rho_1}^D(x, y) = e^{x \cdot \rho_1} G_{\rho_1}^D(x, y)$  and  $G_{\rho_1}^D(x, y)$  is a solution of the following integral equation of the first kind :

$$(12) \quad -G_{\rho_1}(x) = \int_{\partial\Omega} G_{\rho_1}(x, y) \frac{\partial G_{\rho_1}^D}{\partial \nu(y)}(x, y) ds(y).$$

Here, the function  $G_{\rho_1}$  is given by

$$G_{\rho_1}(x) = \int_{\mathbf{R}^3} \frac{e^{ix \cdot \xi}}{\xi^2 + 2i\rho_1 \cdot \xi} d\xi.$$

On the other hand, one can use (9) to find that

$$\rho_1 \cdot \xi = i \frac{|l|}{2} \left[ |\xi| \cos(\widehat{\rho_1, \xi}) - \frac{(-k \cdot \xi + i\eta \cdot \xi)}{|l|} \right].$$

Therefore, to continue with the proof we may insert the last formula into equation (12) and we follow the convenable procedure.  $\square$

Applying Lemma 2 to the results found in Proposition 2, we can prove the following reconstruction formula.

**Theorem 1** *Let  $\mathbf{n}_0 \in L^\infty(\Omega)$  be a given function. Assume that 0 is not a Dirichlet eigenvalue of  $(\frac{1}{\omega^2} \Delta + \mathbf{n})$  in  $\Omega$  and  $\mathbf{n} = \mathbf{n}_0$  almost everywhere in a neighborhood of  $\partial\Omega$ . Then*

$$(\widehat{\mathbf{n} - \mathbf{n}_0})(-k) = -\frac{\sqrt{2}}{\omega^2} \int_{\Gamma} \frac{1}{|\eta|^2} (\eta \cdot \nu(x)|_{\Gamma})^3 e^{ik \cdot x} ds(x),$$

where  $\eta, k \in \mathbf{R}^3$  such that  $\eta \cdot k = 0$ .

Now, we apply Proposition 2 and the asymptotic result given in [2] for identifying efficiently the locations  $\{z_j\}_{j=1}^m$  of the small inhomogeneities  $\mathcal{B}_\alpha$  from the knowledge of the difference between the local DN maps  $\Lambda_{\mathbf{n}_\alpha} - \Lambda_{\mathbf{n}}$  on  $\Gamma$ . The following result holds.

**Theorem 2** *Suppose that we have (1), let  $\mathbf{n} \in C^0(\Omega)$ ,  $\mathbf{n}_\alpha$  be given by (2). Assume that 0 is not a Dirichlet eigenvalue of  $(\frac{1}{\omega^2} \Delta + \mathbf{n}_\alpha)$  in  $\Omega$  and  $\mathbf{n}_\alpha = \mathbf{n}$  almost everywhere in a neighborhood of  $\partial\Omega$  (for  $\alpha$  sufficiently small). Let  $k, l, \rho_1$  and  $\rho_2$  be as in (9). Then, the following identification holds:*

$$\begin{aligned} (\widehat{\mathbf{n}_\alpha - \mathbf{n}})(-k) &= \frac{1}{\omega^2} \lim_{|l| \rightarrow +\infty} \int_{\Gamma} (\Lambda_{\mathbf{n}_\alpha} + N_{\rho_1})^{-1} (\rho_1 \cdot \nu(x) e^{x \cdot \rho_1}|_{\Gamma}) (\Lambda_{\mathbf{n}_\alpha} - \\ &\Lambda_{\mathbf{n}}) (\Lambda_{\mathbf{n}} + N_{\rho_2})^{-1} (\rho_2 \cdot \nu(x) e^{x \cdot \rho_2}|_{\Gamma}) ds(x) = \frac{\alpha^3}{\omega^2} \sum_{j=1}^m (\mathbf{n}(z_j) - \mathbf{n}_j(z_j)) |B_j| e^{ik \cdot z_j} + o(\alpha^3). \end{aligned}$$

As done in Theorem 1, we can deduce the following more explicit result.

**Corollary 1** *Suppose that we have the hypothesis of Theorem 2. Then, the following identification holds:*

$$-\sqrt{2} \int_{\Gamma} \frac{1}{|\eta|^2} (\eta \cdot \nu(x)|_{\Gamma})^3 e^{ik \cdot x} ds(x) = \alpha^3 \sum_{j=1}^m (\mathbf{n}(z_j) - \mathbf{n}_j(z_j)) |B_j| e^{ik \cdot z_j} + o(\alpha^3),$$

where  $\eta, k \in \mathbf{R}^3$  such that  $\eta \cdot k = 0$ .

By neglecting the remainders  $o(\alpha^3)$  in Corollary 1, the locations  $\{z_j\}_{j=1}^m$  are obtained as supports of the inverse Fourier transform of

$$-\sqrt{2} \int_{\Gamma} \frac{1}{|\eta|^2} (\eta \cdot \nu(x)|_{\Gamma})^3 e^{ik \cdot x} ds(x).$$

If, we get the points  $\{z_j\}_{j=1}^m$ , the values  $\{\mathbf{n}_j(z_j)\}_{j=1}^m$  could be obtained by solving a linear system arising from Corollary 1.

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